Notes on Dirichlet L-functions

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1 Bernoulli Numbers and Bernoulli Polynomials

Our investigation of the Bernoulli numbers begins with the following problem. Given an integer $n \geq 0$, we wish to find a polynomial $B_n(x)$ such that for all real numbers a

$$a^n = \int_a^{a+1} B_n(x)dx$$

It is maybe not immediately clear that such polynomials exist but in fact they are not that difficult to construct. We begin by differentiating both sides of the above equation with respect to a and obtain

$$na^{n-1} = B_n(a+1) - B_n(a)$$

Assuming that $n > 0$ we see that

$$\int_a^{a+1} nB_{n-1}(x)dx = B_n(a+1) - B_n(a)$$

This implies that $B_n'(x) = nB_{n-1}(x)$. Using in addition the fact that for $n > 0$,

$$0 = 0^n = \int_0^1 B_n(x)dx$$

we see that given $B_{n-1}(x)$, $B_n(x)$ is uniquely determined by the conditions $B_n'(x) = nB_{n-1}(x)$ and $\int_0^1 B_n(x)dx = 0$. Combining this with the observation that $B_0(x) = 1$ we can inductively construct the Bernoulli polynomials, which are the unique polynomials satisfing the desired condition.

The first few Bernoulli polynomials are $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$, etc. The numerator and denominator of the coefficients of the Bernoulli polynomials grow quite rapidly. Nonetheless, the recursive relation satified by the Bernoulli polynomials makes them relatively easy to compute.

One interesting property of the Bernoulli polynomials is the following. Note that

$$a^n+(a+1)^n+...+(a+k)^n = \int_a^{a+k+1} B_n(x)dx = \frac{1}{n+1}(B_{n+1}(a+k+1)-B_{n+1}(a))$$

Plugging in $a = 0$ we see that

$$1^n + 2^n + ... + k^n = \frac{1}{n+1}(B_{n+1}(k+1) - B_{n+1}(0))$$

Letting $n = 1$, we obtain, for example, the well-known formula that

$$1 + 2 + ... + k = \frac{1}{2}((k+1)^2 - (k+1) + \frac{1}{6} - \frac{1}{6}) = \frac{(k^2 + k)}{2}$$

We can compute $B_3(x) = x^3 - (3/2)x^2 + (1/2)x$ to obtain the less obvious formula

$$1 + 4 + 9 + ... + k^2 = \frac{1}{3}((k+1)^3 - \frac{3}{2}(k+1)^2 + \frac{1}{2}(k+1)) = \frac{(k(2k+1)(k+1))}{6}$$

and this process can clearly be generalized.

We now define $B_n(0) = B_n$ to be the n-th Bernoulli number. Notice that

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}$$

This can be seen by induction as follows. It is clearly true for $n = 0$. Now assume it is true for n, then $B'_{n+1}(x) = (n+1)B_n(x) = \sum_{i=0}^{n}(n+1)\binom{n}{i}B_i x^{n-i}$. Additionally, B_{n+1} is by definition the constant term of $B_{n+1}(x)$. So we compute

$$B_{n+1}(x) = \sum_{i=0}^{n} \frac{n+1}{n-i+1}\binom{n}{i} B_i x^{n-i+1} + B_{n+1}$$

Noticing that $\frac{n+1}{n-i+1}\binom{n}{i} = \binom{n+1}{i}$ we have

$$B_{n+1}(x) = \sum_{i=0}^{n+1} \binom{n+1}{i} B_i x^{n+1-i}$$

as desired.

Now assume that $n > 1$. Then notice that since the integral of $B_{n-1}(x)$ on the interval $[0, 1]$ is 0 and $B'_n(x) = B_{n-1}(x)$ we must have $B_n = B_n(0) = B_n(1)$. In light of the previous formula for $B_n(x)$ we obtain the following relation between the Bernoulli numbers.

$$B_n = \sum_{i=0}^{n} \binom{n}{i} B_i$$

or

$$0 = \sum_{i=0}^{n-1} \binom{n}{i} B_i$$

for $n > 1$. If $n = 1$, then the corresponding relation is simply $B_0 = 1$. This could also be used to compute the Bernoulli numbers.

Our next goal is to derive the exponential generating function for the Bernoulli numbers and in the process to prove that $B_n = 0$ for odd $n > 1$. We proceed as follows. Define

$$f(t,x) = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n$$

Taking derivatives with respect to t we see that

$$\frac{df}{dt} = \sum_{n=1}^{\infty} \frac{B_{n-1}(t)}{(n-1)!} x^n = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^{n+1} = xf(t,x)$$

By solving this differential equation we see that $f(t,x) = g(x)e^{xt}$ for some function $g(x)$. Now we have additionally that

$$\int_0^1 f(t,x)dt = \sum_{n=0}^{\infty} \int_0^1 \frac{B_n(t)}{n!} x^n = 1$$

for all x, since all of the integrals in above sum are 0 except for the first one by the recursive relation that the Bernoulli polynomials satisfy. This permits us to solve for $g(x)$. Namely we have

$$1 = \int_0^1 g(x)e^{xt}dt = g(x)\int_0^1 e^{xt} = g(x)\frac{e^x - 1}{x}$$

from which we obtain that $g(x) = \frac{x}{e^x - 1}$. Combining this we see that

$$f(t,x) = \frac{xe^{tx}}{e^x - 1}$$

and letting $t = 0$ we have

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}$$

Now we wish to show that $B_n = 0$ for odd $n > 1$. Since we know that $\frac{B_1}{1!} = -\frac{1}{2}$ we must simply show that $f(0,x) + \frac{x}{2}$ is an even function of x. But this is simple

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{2x + xe^x - x}{2e^x - 2} = \frac{x(e^x + 1)}{2(e^x - 1)}$$

Replacing x by $-x$ we obtain

$$\frac{-x(e^{-x} + 1)}{2(e^{-x} - 1)} = \frac{-x(1 + e^x)}{2(1 - e^x)} = \frac{x(e^x + 1)}{2(e^x - 1)}$$

so $f(0,x)+\frac{x}{2}$ is an even function and thus $B_n = 0$ for odd $n > 1$ as desired.

One last remark in this section is the following. If we instead decided to find polynomials such that

$$a^n = \int_a^{a+h} P_n(x)dx$$

for any $h \neq 0$ we could proceed as follows. Consider

$$(ha)^n = \int_{ha}^{h(a+1)} P_n(x)dx$$

Now make the change of variables $hy = x$. Then we see that

$$(ha)^n = \int_a^{a+1} P_n(hy)hdy$$

so that

$$a^n = \int_a^{a+1} \frac{1}{h^{n-1}} P_n(hy)dy$$

Consequently, by the uniqueness of the Bernoulli polynomials we must have that $B_n(x) = \frac{1}{h^{n-1}} P_n(hx)$ or $P_n(x) = h^{n-1} B_n(\frac{x}{h})$. So these polynomials can be expressed simply in terms of the Bernoulli polynomials.

If we let $h = \frac{1}{k}$ for some natural number k, then we could have solved the preceding problem differently, by setting $P_n(x) = \sum_{j=0}^{k-1} B_n(x + \frac{j}{k})$. One can easily see that

$$\int_a^{a+\frac{1}{k}} P_n(x)dx = \int_a^{a+1} B_n(x)dx$$

for each n so we must have

$$(\frac{1}{k})^{n-1} B_n(kx) = \sum_{j=0}^{k-1} B_n(x + \frac{j}{k})$$

or

$$B_n(kx) = k^{n-1} \sum_{j=0}^{k-1} B_n(x + \frac{j}{k})$$

To conclude the section on Bernoulli polynomials, I will say that each of the properties derived here is interesting and important for the further application of Bernoulli polynomials. Also note that much of this section can be reformulated in the language of linear algebra. We consider the vectorspace $\mathbb{R}[x]$ of all polynomials with real coefficients and analyse the linear map $T : \mathbb{R}[x] \to \mathbb{R}[x]$ which sends a polynomial $p(x)$ to the polynomial $h(y) = \int_y^{y+1} p(x)dx$. It is a simple matter to verify that this is a linear map such that $deg(p(x)) = deg(T(p(x)))$. This implies that T is a bijection and so has an inverse T^{-1}. Then $B_n(x) = T^{-1}(x^n)$.

2 L-functions

2.1 Characters

Let A be a finite abelian group and consider the group $Hom(A, \mathbb{C}^\times) \cong A$ of group homomorphisms from A to the multiplicative group of complex numbers. This is called the group of characters.

Lemma 2.1 $Hom(A, \mathbb{C}^\times) \cong A$

Proof Since A is a finite abelian group, we know that A is a product of cyclic groups. Since $Hom(X \times Y, Z) \cong Hom(X, Z) \times Hom(Y, Z)$ we must only show that the lemma holds for a finite cyclic group. So assume that A is generated by x and $x^n = 1$. Then any $\chi \in Hom(A, \mathbb{C}^\times)$ is uniquely determined by $\chi(x)$ and $\chi(x)$ must be an n-th root of unity. Hence $Hom(A, \mathbb{C}^\times)$ is isomorphic to the group of n-th roots of unity which is cyclic of order n. This completes the proof.

The next two lemmas relate the characters to the additive structure of $\mathbb{C}$.

Lemma 2.2

$$\sum_{x \in A} \chi(x) = \begin{cases} 0 & \chi \neq 1 \\ |A| & \chi = 1 \end{cases}$$

Proof If $\chi = 1$, the sum is $\sum_{x \in A} 1 = |A|$.

Now assume that $\chi \neq 1$ and let $y \in A$ such that $\chi(y) \neq 1$. Then

$$\sum_{x \in A} \chi(x) = \sum_{x \in A} \chi(yx) = \chi(y) \sum_{x \in A} \chi(x)$$

so $(1 - \chi(y)) \sum_{x \in A} \chi(x) = 0$. As $\chi(y) \neq 1$ we must have $\sum_{x \in A} \chi(x) = 0$.

Lemma 2.3

$$\sum_{\chi \in Hom(A, \mathbb{C}^\times)} \chi(x) = \begin{cases} 0 & x \neq 1 \\ |A| & x = 1 \end{cases}$$

Proof If $x = 1$, the sum is $\sum_{\chi \in Hom(A, \mathbb{C}^\times)} 1 = |A|$.

Now assume that $x \neq 1$. If $\chi(x) = 1$ for all $\chi \in Hom(A, \mathbb{C}^\times)$, then $Hom(A, \mathbb{C}^\times) \cong Hom(A/(x), \mathbb{C}^\times)$ (here (x) is the cyclic subgroup generated by x). But then by the first lemma of this section $A \cong A/(x)$ which is impossible since the groups have different orders.

So pick a $\psi \in Hom(A, \mathbb{C}^\times)$ such that $\psi(x) \neq 1$. Then

$$\sum_{\chi \in Hom(A, \mathbb{C}^\times)} \chi(x) = \sum_{\chi \in Hom(A, \mathbb{C}^\times)} \psi\chi(x) = \psi(x) \sum_{\chi \in Hom(A, \mathbb{C}^\times)} \chi(x)$$

so $(1 - \psi(x)) \sum_{\chi \in Hom(A, \mathbb{C}^\times)} \chi(x) = 0$ and since $\psi(x) \neq 1$ we conclude that $\sum_{\chi \in Hom(A, \mathbb{C}^\times)} \chi(x) = 0$.

In the following we consider the case where $A = (\mathbb{Z}/n\mathbb{Z})^\times$ for some positive integer n. Given a character χ of A we extend $\chi : \mathbb{Z} \to \mathbb{C}$ by setting $\chi(a) = 0$ if $(a, n) \neq 1$. Then χ satisfies $\chi(ab) = \chi(a)\chi(b)$ for any two integers a, b.

2.2 Diriclet Series

In this section we define the main object of study, the Diriclet L-series.

Definition Let χ be the extension of a character $(\mathbb{Z}/n\mathbb{Z})^\times$ to $\mathbb{Z}$ by setting $\chi(a) = 0$ if $(a, n) > 1$. Define the Dirichlet L-function of the character to be

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

The goal of the remainder of this chapter will be to derive as many properties of L-functions as possible. We begin with a very simple lemma.

Lemma 2.4 *The above sum converges absolutely for $Re(s) > 1$.*

Proof This statement follows since $|\chi n/n^s| \leq |1/n^{Re(s)}|$ and thus the above series converges absolutely by comparison with $\sum_{n=1}^{\infty} 1/n^{Re(s)}$ for $Re(s) > 1$.

The above proof also shows that for any $\delta > 0$, the above series converges absolutely and uniformly for $Re(s) \geq 1 + \delta$ which implies that the series converges to an analytic function in the half-plane $Re(s) > 1$.

We will proceed to derive a formula for the analytic continuation of $L(\chi, s)$ to the entire complex plane. In order to do this we will introduce the partial zeta functions $\zeta(s, n; r)$, analytically continue them and write our original L-series in terms of the partial zeta functions.

2.3 Analytic Continuation of Partial Zeta Functions

In this section we define and compute the analytic continuation of the partial zeta functions.

Definition Let $n \in \mathbb{N}$ and $1 \leq r \leq n$. Then we define

$$\zeta(s, n; r) = \sum_{k=1}^{\infty} \frac{1}{(k - \frac{n-r}{n})^s}$$

.

Notice that if $n = r$, we obtain the familiar Riemann zeta function. The reason why the partial zeta functions are useful is that we can analytically continue them using the Euler-Maclauren summation formula.

To obtain the Euler-Maclauren formula, consider a smooth function $f : \mathbb{R} \to \mathbb{C}$ such that $\int_1^\infty |f(x)|dx < \infty$ and $\sum_{n=1}^{\infty} |f(n)| < \infty$. We wish to relate $\sum_{n=1}^{\infty} f(n)$ to $\int_1^\infty f(x)dx$. To this end, rewrite

$$\int_1^\infty f(x)dx = \sum_{n=1}^{\infty} \int_n^{n+1} B_0(x - n) f(x)dx$$

and integrate each of the terms in the sum by parts to obtain

$$\int_1^\infty f(x)dx = \sum_{n=1}^{\infty} B_1(1) f(n+1) - B_1(0) f(n) - \sum_{n=1}^{\infty} \int_n^{n+1} B_1(x - n) f'(x)dx$$

Now the first sum $\sum_{n=1}^{\infty} B_1(1)f(n+1) - B_1(0)f(n)$ can be simplified since $B_1(1) = \frac{1}{2}$ and $B_1(0) = -\frac{1}{2}$ so that

$$\sum_{n=1}^{\infty} B_1(1)f(n+1) - B_1(0)f(n) = -\frac{f(1)}{2} + \sum_{n=1}^{\infty} f(n)$$

We deal with the second term $\sum_{n=1}^{\infty} \int_n^{n+1} B_1(x-n)f'(x)dx$ by integrating by parts $2l$ times to obtain

$$\sum_{n=1}^{\infty} \int_n^{n+1} B_1(x-n)f'(x)dx = \sum_{k=1}^{l} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1)$$

$$-\sum_{n=1}^{\infty} \int_n^{n+1} \frac{B_{2l+1}}{(2l+1)!}(x-n)f^{(2l)}(x)dx$$

This works out since $B_k(x)' = kB_{k-1}(x)$ and $B_k(0) = B_k(1) = B_k$ for $k > 1$ so the sum which shows up when integrating by parts telescopes except for the first term. The odd Bernoulli numbers are 0 so these terms are omitted from the above sum.

Rearranging the above we obtain

$$\sum_{n=1}^{\infty} f(n) = \int_1^{\infty} f(x)dx + \frac{f(1)}{2} - \sum_{k=1}^{l} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1)$$

$$+\sum_{n=1}^{\infty} \int_n^{n+1} \frac{B_{2l+1}}{(2l+1)!}(x-n)f^{(2l)}(x)dx$$

which holds for any $l \in \mathbb{N}$. This looks like a complicated formula, but it is very useful for analytically continuing functions which are expressed as convergent series. This is especially true if higher derivatives of f are more “well-behaved” than f.

How this works will become clear when applying this to the function $f(x) = \frac{1}{(x-\frac{n-r}{n})^s}$ to analytically continue $\zeta(s,n;r)$. We see that for $Re(s) > 1$,

$$\zeta(s,n;r) = \int_1^{\infty} \frac{1}{(x-\frac{n-r}{n})^s} dx + \frac{n^s}{2r^s}$$

$$+\sum_{k=1}^{l} \frac{B_{2k}}{(2k)!} \frac{s(s+1)(s+2)...(s+2k-2)n^{s+(2k-1)}}{r^{s+(2k-1)}}$$

$$+s(s+1)(s+2)...(s+2l-1)\sum_{n=1}^{\infty} \int_n^{n+1} \frac{B_{2l+1}}{(2l+1)!}(x-n)\frac{1}{(x-\frac{n-r}{n})^{s+2l}} dx$$

We evaluate the first integral explicitly to obtain

$$\zeta(s,n;r) = \frac{n^{s-1}}{r^{s-1}(1-s)} + \frac{n^s}{2r^s} + \sum_{k=1}^{l} \frac{B_{2k}}{(2k)!} \frac{s(s+1)(s+2)...(s+2k-2)n^{s+(2k-1)}}{r^{s+(2k-1)}}$$

$$+s(s+1)(s+2)...(s+2l-1)\sum_{n=1}^{\infty} \int_n^{n+1} \frac{B_{2l+1}}{(2l+1)!}(x-n)\frac{1}{(x-\frac{n-r}{n})^{s+2l}} dx$$

The exact details of this formula are not important. What is important is that the last sum is the only "infinite" part of the formula, i.e. everything else can be evaluated explicitly using a finite number of operations. Thus everything except the last sum represents a meromorphic function (defined everywhere except for a pole at $s = 1$(due to the first term). Additionally, since $B_{2l+1}(x)$ is bounded on $[0, 1]$ (it is continuous, being a polynomial) we have that the last sum is bounded by

$$(\sup_{[0,1]} B_{2l+1}(x)) \int_1^\infty \left| \frac{1}{(x - \frac{n-r}{n})^{s+2l}} \right| dx$$

This is finite as long as $Re(s) > 1-2l$ (since $\int_q^\infty x^{-t} dx < \infty$ if $q > 0$ and $t > 1$). Thus by a standard argument the final sum converges uniformly to an analytic function as long as $Re(s) > 1 - 2l$. Since l was arbitrary, this formula gives the analytic continuation of $\zeta(s, n; r)$.

2.4 The Values of L-functions at Non-positive Integers

In this section we use the result of the previous section to analytically continue Dirichlet L-series to the entire complex plane and to evaluate the corresponding L-functions at non-positive integers (by L-function I mean the analytic continuation of an L-series).

Our method for doing this will be to note that the L-series can be written in terms of the partial zeta functions as follows

$$L(\chi, s) = \sum_{k=1}^\infty \frac{\chi(k)}{k^s} = \sum_{r=1}^n \chi(r) \frac{\zeta(s, n; r)}{n^s}$$

where χ is a character of conductor n, i.e. the value of χ depends only on the congruence class mod n. The above formula follows since

$$\chi(r) \frac{\zeta(s, n; r)}{n^s} = \chi(r) \sum_{k=1}^\infty \frac{1}{(n(k-1) + r)^s} = \sum_{k \equiv r(n)} \frac{\chi(k)}{k^s}$$

Now we see that the analytic continuation of the partial zeta functions derived in the preceding section provide the analytic continuation of L-series.

Moreover, the values of the L-functions at non-positive integers can be determined from the formula for the continuation of the partial zeta functions. We recall from last section that

$$\zeta(s, n; r) = \frac{n^{s-1}}{r^{s-1}(1-s)} + \frac{n^s}{2r^s} + \sum_{k=1}^l \frac{B_{2k}}{(2k)!} \frac{s(s+1)(s+2)...(s+2k-2)n^{s+(2k-1)}}{r^{s+(2k-1)}}$$

$$+ s(s+1)(s+2)...(s+2l-1) \sum_{n=1}^\infty \int_n^{n+1} \frac{B_{2l+1}}{(2l+1)!}(x-n) \frac{1}{(x - \frac{n-r}{n})^{s+2l}} dx$$

and that if s is a non-positive integer and l is large enough, then the last term in the sum is 0 so we obtain a finite expression for $\zeta(s, n; r)$. In particular, we have

$$\zeta(-m, n; r) = \frac{1}{m+1} \left(\frac{n}{r}\right)^{-(m+1)} + \frac{1}{2} \left(\frac{n}{r}\right)^{-m} + \frac{1}{m+1} \sum_{k=1}^l B_{2k} \binom{m+1}{2k} \left(\frac{n}{r}\right)^{-m+2k-1}$$

Where l is large enough so that $2l-1 \geq m$. Noticing that $B_0 = 1$, $B_1 = \frac{1}{2}$, and odd Bernoulli numbers vanish, we rewrite the above as

$$\zeta(-m, n; r) = \frac{1}{m+1} \sum_{k=0}^{m+1} B_k \binom{m+1}{k} \left(\frac{r}{n}\right)^{m+1-k}$$

Recalling the relationship between the Bernoulli numbers and Bernoulli polynomials we see that

$$\zeta(-m, n; r) = \frac{1}{m+1} B_{m+1}\left(\frac{r}{n}\right)$$

In particular, if $r = n$, we see that

$$\zeta(-m) = \frac{1}{m+1} B_{m+1}(1) = \frac{B_{m+1}}{m+1}$$

Finally we note that since

$$L(\chi, s) = \sum_{r=1}^{n} \chi(r) \frac{\zeta(s, n; r)}{n^s}$$

we have that

$$L(\chi, -m) = \frac{1}{m+1} \sum_{r=1}^{n} \chi(r) n^m B_{m+1}\left(\frac{r}{n}\right)$$

Now we make the following definition

Definition Let χ be a Dirichlet character of conductor n. Then define the generalized Bernoulli numbers $B_{\chi,k}$ as

$$B_{\chi,k} = n^{k-1} \sum_{r=1}^{n} B_k\left(\frac{r}{n}\right)$$

Using this new notation we have that

$$L(\chi, -m) = \frac{B_{\chi,m+1}}{m+1}$$

Notice that since the Bernoulli polynomials have rational coefficients, we see that $L(\chi, -m) \in \mathbb{Q}(\zeta_n)$ (here n is the conductor of m). Additionally, the above formulas give us an explicit way of representing $L(\chi, -m)$ in $\mathbb{Q}(\zeta_n)$.